

# Analytical estimation of the corrections to the apparent value of the cosmological constant due to large scale structure

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## Abstract

We compute the Taylor expansion for the luminosity distance for an observer at the center of a spherically symmetric matter inhomogeneity with a non vanishing cosmological constant. Defining apparent the value of the cosmological constant estimated by an observer who observes the same luminosity distance but assumes homogeneity in its estimation of the cosmological parameters, we derive a relation between the apparent and the true value, i.e. the one obtained taking into account the inhomogeneity. We can consider the assumption of being at the center of a spherically symmetric inhomogeneous matter distribution equivalent to calculating the monopole contribution of the large scale structure around us, which we expect to be the dominant one, because of other observations such as the cosmic microwave background radiation supporting a high level of isotropy.

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## I. INTRODUCTION

Luminosity distance measurements [1–6] and the WMAP measurements [7, 8] of cosmic microwave background (CMB) interpreted in the context of standard FLRW cosmological models strongly support a dominant dark energy component.

As an alternative to dark energy, some of attention has been devoted to the possibility of interpreting the same observational data with inhomogeneous cosmological model [9, 10, 12–29]

Our approach will be different, since we will consider a Universe with a cosmological constant and some local large scale inhomogeneity modeled by a  $\Lambda$ LTB solution [30]. For simplicity we will also assume that we are located at its center. In this regard this can be considered a first attempt to model local large scale inhomogeneities in the presence of the cosmological constant or, more in general, dark energy. Given the spherical symmetry of the LTB solution and the assumption to be located at the center our calculation can be interpreted as the monopole contribution of the large inhomogeneities which surround us. Since we know from other observations such as CMB radiation that the Universe appears to be highly isotropic, we can safely assume that the monopole contribution we calculate should also be the dominant one, making our results even more relevant. We first calculate the null radial geodesics for a central observer and then use it to obtain the luminosity distance in a LTB model with non vanishing cosmological constant. We then compare it to that of  $\Lambda$ CDM models, finding the relation between the two different cosmological constants appearing in the two models, where we call apparent the one in the  $\Lambda$ CDM and true the one in  $\Lambda$ LTB.

As an application we then calculate the local Taylor expansion of the function defining an LTB model which is able to account for a 5% difference between the apparent and the true value of the cosmological constant.

## II. LTB SOLUTION WITH A COSMOLOGICAL CONSTANT

The LTB solution can be written as [31–33] as

$$ds^2 = -dt^2 + \frac{(R_{,r})^2}{1 + 2E(r)} dr^2 + R^2 d\Omega^2, \quad (1)$$

where  $R$  is a function of the time coordinate  $t$  and the radial coordinate  $r$ ,  $E(r)$  is an arbitrary function of  $r$ , and  $R_{,r} = \partial_r R(t, r)$ . The Einstein equations with dust and a cosmological

constant give

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{2E(r)}{R^2} + \frac{2M(r)}{R^3} + \frac{\Lambda}{3}, \quad (2)$$

$$\rho(t, r) = \frac{2M_{,r}}{R^2 R_{,r}}, \quad (3)$$

with  $M(r)$  being an arbitrary function of  $r$ ,  $\dot{R} = \partial_t R(t, r)$  and  $c = 8\pi G = 1$  is assumed throughout the paper.

Eq.(2) is obtained by integration respect to time of one of the Einstein's field equations, and the function  $M(r)$  plays in fact the role of constant of integration. Since Eq. (2) contains partial derivatives respect to time only, its general solution can be obtained from the FLRW equivalent solution by making every constant in the latter one an arbitrary function of  $r$ .

The general analytical solution for a FLRW model with dust and cosmological constant was obtained by Edwards [34] in terms of elliptic functions. By an appropriate choice of variables and coordinates, we may extend it to the LTB case thanks to the spherical symmetry of both LTB and FLRW models, and to the fact that dust follows geodesics without being affected by adjacent regions. An analytical solution can be found by introducing a new coordinate  $\eta = \eta(t, r)$  and a variable  $a$  by

$$\left(\frac{\partial \eta}{\partial t}\right)_r = \frac{r}{R} \equiv \frac{1}{a}, \quad (4)$$

and new functions by

$$\rho_0(r) \equiv \frac{6M(r)}{r^3}, \quad k(r) \equiv -\frac{2E(r)}{r^2}. \quad (5)$$

Then Eq. (2) becomes

$$\left(\frac{\partial a}{\partial \eta}\right)^2 = -k(r)a^2 + \frac{\rho_0(r)}{3}a + \frac{\Lambda}{3}a^4, \quad (6)$$

where  $a$  is now regarded as a function of  $\eta$  and  $r$ ,  $a = a(\eta, r)$ . It should be noted that the coordinate  $\eta$ , which is a generalization of the conformal time in a homogeneous FLRW universe, has been only implicitly defined by Eq. (4). The actual relation between  $t$  and  $\eta$  can be obtained by integration once  $a(\eta, r)$  is known:

$$t(\eta, r) = \int_0^\eta a(x, r)dx + t_b(r), \quad (7)$$

which can be computed analytically, and involve elliptic integrals of the third kind[35].

The function  $t_b(r)$  plays the role of constant of integration, and is an arbitrary function of  $r$ , sometime called bang function, since by construction at time  $t = t_b(r)$  we have  $a(t_b(r), r) = 0$ ,

and correspond to the fact that the big bang initial singularity can happen at different times at different positions from the center in a LTb space. In the rest of this paper we will assume homogeneous bang, i.e. we will set

$$t_b(r) = 0. \quad (8)$$

Inspired by the construction of the solution for the FLRW case get:

$$a(\eta, r) = \frac{\rho_0(r)}{3\phi\left(\frac{\eta}{2}; g_2(r), g_3(r)\right) + k(r)}, \quad (9)$$

where  $\phi(x; g_2, g_3)$  is the Weierstrass elliptic function satisfying the differential equation

$$\left(\frac{d\phi}{dx}\right)^2 = 4\phi^3 - g_2\phi - g_3, \quad (10)$$

and

$$\alpha = \rho_0(r), \quad g_2 = \frac{4}{3}k(r)^2, \quad g_3 = \frac{4}{27}\left(2k(r)^3 - \Lambda\rho_0(r)^2\right). \quad (11)$$

In this paper we will choose the so called FLRW gauge, i.e. the coordinate system in which  $\rho_0(r)$  is constant.

It is convenient to re-write the solution in terms of dimensionless quantities [37]:

$$k(r) = (a_0 H_0)^2 K(r), \quad (12)$$

$$\eta = T(a_0 H_0)^{-1}, \quad (13)$$

$$\rho_0 = 3\Omega_m^0 a_0^3 H_0^2, \quad (14)$$

$$\Lambda = 3\Omega_\Lambda H_0^2, \quad (15)$$

$$a(\eta, r) = a(T(a_0 H_0)^{-1}, r) = \tilde{a}(T, r), \quad (16)$$

$$(17)$$

to obtain

$$\tilde{a}(T, r) = \frac{3a_0\Omega_M^0}{K(r) + 12\tilde{\phi}(T, g_2(r), g_3(r))}, \quad (18)$$

$$g_2(r) = \frac{K(r)^2}{12}, \quad (19)$$

$$g_3(r) = \frac{1}{432}\left(2K(r)^3 - 27\Omega_\Lambda(\Omega_M^0)^2\right). \quad (20)$$

or after multiplying every term by  $(a_0 H_0)^2$  and using the original dimensionful quantities  $\eta, k(r), \rho_0$

$$a(\eta, r) = \frac{\rho_0}{k(r) + 12\phi(\eta, g_2(r), g_3(r))} = \tilde{a}(T, r), \quad (21)$$

$$\phi(\eta, r) = \tilde{\phi}(\eta(a_0 H_0), r)(a_0 H_0)^2 = \tilde{\phi}(T, r)(a_0 H_0)^2. \quad (22)$$

In this form  $H_0$  is an arbitrary scale which we can set equal to the observed value and to the  $H_0^{LTB}$  by appropriately setting the value of  $T_0$  as explained in more details in [37].

### III. GEODESIC EQUATIONS AND LUMINOSITY DISTANCE

We use the same method developed in [36] to solve the null geodesic equation written in terms of the coordinates  $(\eta, r)$ . Rather than integrating differential equations numerically, we perform a local expansion of the solution around  $z = 0$  corresponding to the point  $(t_0, 0)$ , or equivalently  $(\eta_0, 0)$ , where  $t_0 = t(\eta_0, 0)$ .

The luminosity distance for a central observer in the LTB space-time as a function of the redshift  $z$  is

$$D_L(z) = (1+z)^2 R(t(z), r(z)) = (1+z)^2 r(z) a(\eta(z), r(z)), \quad (23)$$

where  $(t(z), r(z))$  or  $(\eta(z), r(z))$  is the solution of the radial geodesic equation as a function of  $z$ . The equation for radial null geodesics can be easily obtained in the coordinates  $(t, r)$

$$\frac{dt}{dr} = -\frac{R_{,r}(t, r)}{\sqrt{1 + 2E(r)}}. \quad (24)$$

where  $t = T(r)$  is the time coordinate along the null radial geodesic as a function of the coordinate  $r$ . In terms of  $z$ , Eq. (24) takes the form [38]:

$$\begin{aligned} \frac{dr}{dz} &= \frac{\sqrt{1 + 2E(r(z))}}{(1+z)\dot{R}_r[r(z), t(z)]}, \\ \frac{dt}{dz} &= -\frac{R_{,r}[r(z), t(z)]}{(1+z)\dot{R}_r[r(z), t(z)]}. \end{aligned} \quad (25)$$

These equations are derived from the definition of redshift and by following the evolution of a short time interval along the null geodesic  $T(r)$ . It can be shown [36] that in the coordinates  $(\eta, r)$  eqs. (25) take the form:

$$\frac{d\eta}{dz} = -\frac{\partial_r t(\eta, r) + F(\eta, r)}{(1+z)\partial_\eta F(\eta, r)} \equiv p(\eta, r), \quad (26)$$

$$\frac{dr}{dz} = \frac{a(\eta, r)}{(1+z)\partial_\eta F(\eta, r)} \equiv q(\eta, r), \quad (27)$$

where

$$F(\eta, r) \equiv \frac{R_{,r}}{\sqrt{1+2E(r)}} = \frac{1}{\sqrt{1-k(r)r^2}} \left[ \partial_r(a(\eta, r)r) - a^{-1} \partial_\eta(a(\eta, r)r) \partial_r t(\eta, r) \right]. \quad (28)$$

It is important to observe that the functions  $p, q, F$  have explicit analytical forms, making it particularly useful to derive analytical results.

#### IV. FORMULA FOR THE LUMINOSITY DISTANCE

In order to obtain the redshift expansion of the luminosity distance we need to use the following:

$$k(r) = (a_0 H_0)^2 K(r) = K_0 + K_1 r + K_2 r^2 + \dots \quad (29)$$

$$t(\eta, r) = b_0(\eta) + b_1(\eta)r + b_2(\eta)r^2 + \dots \quad (30)$$

We will use eq.(7) to obtain the above expansion for  $t(\eta, r)$  from the exact solution for  $a(\eta, r)$ . Following the same approach given in [30], we can find a local Taylor expansion in red-shift for the geodesics equations, and then calculate the luminosity distance. The general expression is rather cumbersome, so will report here only the result assuming  $K_0 = 0$ , which is still showing the general nature of the effect. We will expand the solution of the geodesic equations according to:

$$r(z) = r_1 z + r_2 z^2 + \dots \quad (31)$$

$$\eta(z) = \eta_1 z + \eta_2 z^2 + \dots \quad (32)$$

$$K(z) = K_1 z + K_2 z^2 + \dots \quad (33)$$

$$(34)$$

After substituting the above expansion in the geodesics equation we can reduce the solution of the differential equations to the solution of a system of algebraic equations for the expansion coefficient.

For the geodesics equations we get:

$$\begin{aligned} \eta_1 &= -\frac{K_1(T_0 - 1)T_0 + 3\Omega_M}{3a_0 H_0 \Omega_M}, \\ \eta_2 &= \frac{1}{36a_0 H_0 \Omega_\Lambda \Omega_M^2} \left[ 3\Omega_\Lambda \Omega_M (9\Omega_M^2 - 4K_2(-1 + T_0)T_0) + 3K_1 \Omega_\Lambda \Omega_M (-4 + (4 - 9\Omega_M)T_0 + \right. \\ &\quad \left. + (-4 + 9\Omega_M)T_0^2) + K_1^2 T_0 (2\Omega_\Lambda (2 + (-4 + 3\Omega_M)T_0 - 6(-1 + \Omega_M)T_0^2 + 3(-1 + \Omega_M)T_0^3) \right] \end{aligned} \quad (35)$$

$$-4(-1 + T_0)WZ + \Omega_M(-4 + T_0 + 3T_0WZ)) \Big], \quad (36)$$

$$r_1 = \frac{1}{a_0 H_0}, \quad (37)$$

$$r_2 = -\frac{1}{12a_0 H_0 \Omega_M} \left[ (9\Omega_M^2 + (-4 + (4 - 6\Omega_M)T_0 + (-4 + 6\Omega_M)T_0^2)) \right], \quad (38)$$

$$r_3 = \frac{K_1^2}{72a_0 H_0 \Omega_\Lambda \Omega_M^2} \left[ 2\Omega_\Lambda (6(3\Omega_M^2 - 4\Omega_M + 1)T_0^4 - 12(3\Omega_M^2 - 4\Omega_M + 1)T_0^3 + 3(6\Omega_M^2 - 13\Omega_M + 4)T_0^2 + \right. \\ \left. + 2(9\Omega_M - 4)T_0 + 4) + 3\Omega_M^2 T_0 (3T_0 \zeta_0 + T_0 - 4) - 2\Omega_M (9T_0^2 \zeta_0 + T_0^2 - 6T_0 \zeta_0 - 4T_0 + \right. \\ \left. 6\zeta_0 - 2) + 8(T_0^2 - T_0 + 1)\zeta_0) + 12K_1 \Omega_\Lambda \Omega_M^2 ((9\Omega_M - 8)T_0^2 + (8 - 9\Omega_M)T_0 - 5) + \right. \\ \left. + 3\Omega_\Lambda \Omega_M (K_2((8 - 12\Omega_M)T_0^2 + 4(3\Omega_M - 2)T_0 + 8) + 3(9\Omega_M - 4)\Omega_M^2)) \right]. \quad (39)$$

After substituting in the formula for the luminosity distance and expanding we finally get:

$$D_L^{ALTB}(z) = (1 + z)^2 r(z) a^{ALTB}(\eta(z), r(z)) = D_1^{ALTB} z + D_2^{ALTB} z^2 + D_3^{ALTB} z^3 + .. \quad (40)$$

$$D_1^{ALTB} = \frac{1}{H_0}, \quad (41)$$

$$D_2^{ALTB} = -\frac{1}{4H_0} (-4 + 3\Omega_M + 2K_1(-1 + T_0)T_0), \quad (42)$$

$$D_3^{ALTB} = \frac{1}{24H_0 \Omega_\Lambda \Omega_M} \left[ K_1^2 (2\Omega_\Lambda T_0 ((6\Omega_M - 5)T_0^3 - 2(6\Omega_M - 5)T_0^2 + 6(\Omega_M - 1)T_0 + 2) + \right. \\ \left. + \Omega_M T_0 (3T_0 \zeta_0 + T_0 - 4) - 4(T_0^2 \zeta_0 - T_0 \zeta_0 + \zeta_0 - 1)) + \right. \\ \left. + 4K_1 \Omega_\Lambda \Omega_M ((9\Omega_M - 8)T_0^2 + (8 - 9\Omega_M)T_0 - 2) + \right. \\ \left. + 3\Omega_\Lambda \Omega_M (-4K_2(T_0 - 1)T_0 + 9\Omega_M^2 - 10\Omega_M) \right], \quad (43)$$

where we used the Einstein equation at the center ( $\eta = \eta_0, r = 0$ )

$$1 = \Omega_k(0) + \Omega_M + \Omega_\Lambda = -K_0 + \Omega_M + \Omega_\Lambda, \quad (44)$$

$$\Omega_k(r) = -\frac{k(r)}{H_0^2 a_0^2}, \quad (45)$$

$$\Omega_M = \frac{\rho_0}{3H_0^2 a_0^3}, \quad (46)$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}. \quad (47)$$

and  $T_0 = \eta_0(a_0 H_0)$  is determined numerically by imposing the conditions [37]

$$H^{LTB} = \frac{\partial_t a(t, r)}{a(t, r)} = \frac{\partial_\eta a(\eta, r)}{a(\eta, r)^2} = (a_0 H_0) \frac{\tilde{a}'(T, r)}{a(T, r)^2}, \quad (48)$$

$$a(\eta_0, 0) = a_0, \quad (49)$$

$$H^{LTB}(\eta_0, 0) = H_0. \quad (50)$$

In the above formulae we have also defined

$$\zeta_0 = \zeta(\eta_0, g_2(0), g_3(0)), \quad (51)$$

where  $\zeta$  is the Weierstrass Zeta Function satisfying the equation

$$\frac{d\zeta(z, g_2(r), g_3(r))}{dz} = -\phi(z, g_2(r), g_3(r)), \quad (52)$$

The presence of this last function in the formulae obtained above is due to the fact that the function  $t(\eta, r)$  which enters the geodesics equation is the integral of  $a(\eta, r)$ , and since this depends on  $\phi(z)$ , its integral will depend on  $\zeta(z)$ . In the case of a LTB solution without a cosmological constant this integral can be performed without the introduction of a new function, while in this case it requires the introduction of  $\zeta_0$  in the final formula.

The procedure to reduce the analytical formula to this form is rather complicated since it involves to express wherever possible all the intermediate expressions in terms of physically meaningful quantities and to use the properties of the elliptic functions. We give more details about it in the appendix.

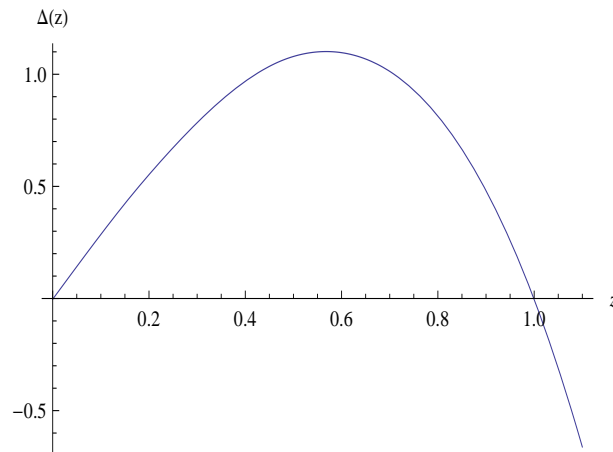


FIG. 1: The percentual error  $\Delta = 100 \frac{D_{Taylor}^{LTB} - D_{Num}^{LTB}}{D_{Taylor}^{LTB}}$  between the numerically computed  $D_{Num}^{LTB}$  and the Taylor third order expansion  $D_{Taylor}^{LTB}$  is plotted as a function of the redshift is plotted for the LTB solution corresponding to  $K(r) = K_1 r + K_2 r^2$ . The coefficients  $K_1, K_2$  are obtained by matching the third order expansion of the luminosity distance of a  $\Lambda$ CDM model using the Taylor expansion obtained in this paper, but with a smaller value of the cosmological constant  $\Omega_{\Lambda}^{LTB} = 95\% \Omega_{\Lambda}^{\Lambda CDM}$ . As it can be seen the Taylor approximation is quite accurate at low red-shift.



## V. CALCULATING $D_L(z)$ FOR $\Lambda$ CDM MODELS.

The metric of a  $\Lambda$ CDM model is the FLRW metric, a special case of LTB solution, where :

$$\rho_0(r) \propto \text{const}, \quad (53)$$

$$k(r) = 0, \quad (54)$$

$$t_b(r) = 0, \quad (55)$$

$$a(t, r) = a(t). \quad (56)$$

We will calculate independently the expansion of the luminosity distance and the redshift spherical shell mass for the case of a flat  $\Lambda$ CDM, to clearly show the meaning of our notation, and in particular the distinction between  $\Omega_\Lambda^{app}$  and  $\Omega_\Lambda^{true}$ . We can also use these formulas to check the results derived before, since in absence of inhomogeneities they should coincide.

One of the Einstein equation can be expressed as:

$$H^{\Lambda CDM}(z) = H_0 \sqrt{(1 - \Omega_\Lambda^{app}) \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda^{app}} = H_0 \sqrt{(1 - \Omega_\Lambda^{app})(1+z)^3 + \Omega_\Lambda^{app}}. \quad (57)$$

We can then calculate the luminosity distance using the following relation, which is only valid assuming flatness:

$$D_L^{\Lambda CDM}(z) = (1+z) \int_0^z \frac{dz'}{H^{\Lambda CDM}(z')} = D_1^{\Lambda CDM} z + D_2^{\Lambda CDM} z^2 + D_3^{\Lambda CDM} z^3 + \dots \quad (58)$$

From which we can get:

$$D_1^{\Lambda CDM} = \frac{1}{H_0}, \quad (59)$$

$$D_2^{\Lambda CDM} = \frac{3\Omega_\Lambda^{app} + 1}{4H_0}. \quad (60)$$

We can check the consistency between these formulae and the ones derived in the case of LTB by setting:

$$K_1 = K_0 = 0, \quad (61)$$

$$(62)$$

which corresponds to the case in which  $\Omega_\Lambda^{app} = \Omega_\Lambda^{true}$ .

## VI. CORRECTION TO THE APPARENT VALUE OF THE COSMOLOGICAL CONSTANT

As explained in [40], the matching of the first two order terms of the Taylor expansion of the luminosity distance of a  $\Lambda$ LTB and a  $\Lambda$ CDM model are sufficient to obtain the leading

order relation between the apparent and true value of the cosmological constant. The matching of higher orders terms gives in fact relations necessary to set the other coefficients of the inhomogeneity profile, but do not affect the relation among  $\Omega^{app}$  and  $\Omega^{true}$ . After calculating the first two terms of the redshift expansion of the luminosity distance for  $\Lambda LTB$  and  $\Lambda CDM$  model we can now look for the  $\Lambda LTB$  models which give the same theoretical prediction the best fit  $\Lambda CDM$  model.

From the above relations we can derive :

$$\Omega_{\Lambda}^{app} = \Omega_{\Lambda}^{true} - \frac{2}{3}K_1(T_0 - 1)T_0, \quad (63)$$

$$K_2 = \frac{1}{12(\Omega_{\Lambda}^t - 1)\Omega_{\Lambda}^t(T_0 - 1)} \left[ K_1(K_1(10\Omega_{\Lambda}^t T_0^4 - 20\Omega_{\Lambda}^t T_0^3 + T_0^2(3\Omega_{\Lambda}^{true}\zeta_0 + 13\Omega_{\Lambda}^{true} + \right. \quad (64)$$

$$\left. + \zeta_0 - 1) - 4T_0(2\Omega_{\Lambda}^{true} + \zeta_0 - 1) + 4(\zeta_0 - 1)) + \right. \quad (65)$$

$$\left. - 4(\Omega_{\Lambda}^{true} - 1)\Omega_{\Lambda}^{true}(3T_0^2 - 3T_0 + 2)) \right].$$

As expected the relation above reduces to

$$\Omega_{\Lambda}^{true} = \Omega_{\Lambda}^{app}, \quad (66)$$

in the limit in which there is no inhomogeneity, i.e. when  $K_1 = 0$ . We should remember that all the formulae above are derived under the assumption that  $K_0 = 0$ , and that the value of  $T_0$  depends on  $\Omega_{\Lambda}^{true}$  and  $K_0$  through the procedure which was explained in the previous sections. A more general formula can be derived also the case in which  $K_0$  is not zero, but given its rather complicated form we do not report it here, since the one derived above already captures the essence of the effect of the large scale inhomogeneity on the apparent value of the cosmological constant. We can see in fact that the correction can be positive or negative depending on the sign of  $K_1$ , which is in qualitative agreement with previous studies of the effects of inhomogeneities on the effective equations of state of a  $\Lambda LTB$  cosmological model [30].

As an application we assume a 5% difference between the apparent and the true value of the cosmological constant and compute  $K_1, K_2$  from the matching of the coefficients of the Taylor expansion of the luminosity distance, obtaining:

$$K_1 = -0.007462, \quad (67)$$

$$K_2 = 0.007037. \quad (68)$$

This gives the  $\Lambda LTB$  model able to mimic locally such a difference in the value of  $\Omega_{\Lambda}^{true}$  and  $\Omega_{\Lambda}^{app}$ .

The accuracy of the formula is then compared to the numerical calculation of the luminosity distance, showing a good agreement.

## VII. CONCLUSIONS

We have derived the correction due to large scale structure to the value of the apparent cosmological constant inferred from low redshift supernovae observations. This analytical calculation shows how the presence of a local inhomogeneity can affect the estimation of the value of cosmological parameters, such as  $\Omega_\Lambda$ . This effects should be properly taken into account both theoretically and observationally. It is important to underline here that we do not need a large void as normally assumed in previous studies of *LTB* models in a cosmological context. Even a small inhomogeneity could in fact be important. The main advantage of the analytical approach we adopted is that it allows to understand the effects in a model independent way, without making any assumption about the nature of the inhomogeneity, which could be either a local under or over density. The formula we derived predicts in fact that the cosmological could be both bigger or smaller than the apparent value obtained by assuming large scale homogeneity. In the future it will also be interesting to extend the same analysis to other observables such as barionic acoustic oscillations (BAO) or the cosmic microwave background radiation (CMBR), and we will report about this in separate papers. Another direction in which the present work could be extended is modeling the local inhomogeneity in a more general way, for example considering not spherically symmetric solutions. From this point of view our calculation could be considered the monopole contribution to the general effect due to a local large scale inhomogeneity of arbitrary shape. Given the high level of isotropy of the Universe shown by other observations such as the CMB radiation, we can expect nevertheless the monopole contribution we calculated to be the dominant one.

### Appendix A: derivation of the analytical formulae

In order to obtain the formula for  $D_L(z)$  in the form in which we reported it in the previous sections, we need to apply several simplifying procedures. From the definition of  $a_0$  and  $H_0$  we can get:

$$\phi_0 = \phi(\eta_0, g_2(0), g_3(0)) = \frac{\rho_0 - a_0 k_0}{3a_0}, \quad (1)$$

$$\phi'_0 = \frac{\phi(\eta, g_2(0), g_3(0))}{d\eta} \Big|_{\eta=\eta_0} = -\frac{2H_0\rho_0}{3}. \quad (2)$$

We can then substitute the above expressions everywhere  $\{\phi_0, \phi'_0\}$  appear, which is the reason why they are not present in the formulae obtained in the previous sections.

Another useful relation is the one which can be obtained from the differential equation defining the Weierstrass elliptic function eq.(10):

$$\tilde{\phi}'_0 = \sqrt{-\frac{K_0^3}{216} - \frac{K_0^2\tilde{\phi}_0}{12} + \frac{\Omega_\Lambda(\Omega_M^0)^2}{16} + 4\tilde{\phi}_0^3}. \quad (3)$$

which can be used for example to obtain  $T_0$  from eq.(49,50), since it allows to determine  $\phi_0$  analytically and then  $T_0$  numerically by solving the equation :

$$\phi(T_0, g_2(0), g_3(0)) = \phi_0. \quad (4)$$

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